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On Bounds for the Norm of Certain Symmetric Projections with a Finite Carrier

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1. INTRODUCTION

Let $C[-1, 1]$ denote the space of real valued and continuous functions in the interval $[-1, 1]$ (endowed with the sup-norm) and let Π_n be the subspace of algebraic polynomials of degree at most n ($n \geqslant 0$). By P we denote a bounded linear map; $P : C[-1, 1] \to \Pi_n$. Additionally we assume that P is idempotent (i.e., $P^2 = P$). Such an operator P will be called a *projection.* The general form of a projection is known (see, e.g., [6]). An important class of projections are the so-called *projections with finite carrier* [3] given by the formula

$$Pf = \sum_{i=0}^{m} f(t_i)\, y_i \tag{1.1}$$

where $f \in C[-1, 1]$, $-1 \leqslant t_0 < t_1 < \cdots < t_m \leqslant 1$, and $y_i \in \Pi_n$ ($i = 0, 1,\ldots, m$; $m \geqslant n$). We say that the projection (1.1) is carried by the nodal set $\{t_i\}$ ($i = 0, 1,\ldots, m$). The condition $P^2 = P$ forces the following equalities for the polynomials y_i (see, e.g., [6])

$$\sum_{i=0}^{m} t_i^k y_i(t) = t^k \qquad (k = 0, 1,\ldots, n). \tag{1.2}$$

The so called *Lebesgue function*

$$\Lambda_P(t) = \sum_{i=0}^{m} | y_i(t)|$$

is connected with the above projection P. It is known that $\| P \| = \| \Lambda_P \|_\infty$.

Let $R : C[-1, 1] \to \Pi_n$ be an arbitrary projection (in particular may be $R = P$). The elementary but important inequality

$$\| f - Rf \|_\infty \leqslant (1 + \| R \|)\, \mathrm{dist}(f, \Pi_n)$$

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holds. From this it is obvious that the accuracy of approximation of f by Rf depends on the norm of the projection R. The quantity bounding from above the error $\| f - Rf \|_\infty$ is small if $\| R \|$ is small. Hence we see that the information on the norm of the projection is very important. It also known [7] that for the above projection R is $\| R \| \geq (2/\pi^2) \log n - O(1)$.

Let Σ denote the class of all projections from $C[-1, 1]$ onto Π_n. The projection $\bar{R} \in \Sigma$ is called a *minimal projection* if the inequality $\| \bar{R} \| \leq \| R \|$ holds for all $R \in \Sigma$.

Let σ be a subclass of Σ containing the projections of the form (1.1) for which the symmetry conditions

$$Pf(x)(t) = Pf(-x)(-t) \qquad (x, t \in [-1, 1]) \tag{1.3}$$

hold. Every such projection will be called a *symmetric projection with finite carrier* (SPFC).

Now we assume that $m = n + 1$.

In this note some bounds for the norm of the minimal SPFC $\bar{P}$ with nodal set the extrema of the Chebyshev polynomial of the first kind are given.

For simplicity set

$$\lambda_n = \frac{1}{n+1} \sum_{i=1}^{n+1} \tan \frac{(2i-1)\pi}{4(n+1)}, \tag{1.4}$$

$$t_i = -\cos \frac{\pi i}{n+1} \qquad (i = 0, 1, \ldots, n+1). \tag{1.5}$$

It is known [1] that

$$\lambda_n = (2/\pi) \log n + O(1). \tag{1.6}$$

The proof of the following theorem is postponed to Section 4.

THEOREM 1.1. *Let n be an even number ($n \geq 0$). Then for the norm of the minimal* SPFC $\bar{P}$ *with nodes* (1.5) *the following inequalities hold*

$$\lambda_n \leq \| \bar{P} \| \leq \lambda_n + 1 \tag{1.7}$$

where λ_n is given by (1.4). *Hence, by virtue of* (1.6), $\| \bar{P} \| \sim (2/\pi) \log n$.

In the proof of this theorem an important role is played by the so-called *operator of de la Vallée Poussin* [4] described in Section 2.

2. THE OPERATOR OF DE LA VALLÉE POUSSIN

For the function $f \in C[-1, 1]$ and $n + 2$ distinct points $\{t_i\}$ ($-1 = t_0 < t_1 < \cdots < t_{n+1} = 1$) we denote by A the operator the value of which is the

nth optimal (in sup-norm) polynomial for the function f on the above set $\{t_i\}$. It is known that A is a projection with value given by the formula

$$Af = \sum_{i=0}^{n+1} f(t_i)\, q_i \tag{2.1}$$

where

$$\begin{aligned} q_i &= z_i - \theta_i r, \\ z_i(t) &= \alpha_i \prod_{\substack{j=0 \\ j\neq i}}^{n+1} (t - t_j), \\ \alpha_i &= \prod_{\substack{j=0 \\ j\neq i}}^{n+1} (t_i - t_j)^{-1}, \\ \theta_i &= (-1)^{n-1} \alpha_i \Big/ \sum_{j=0}^{n+1} |\alpha_j|, \\ r &= \sum_{i=0}^{n+1} (-1)^i z_i \qquad (i = 0, 1,..., n+1) \end{aligned} \tag{2.2}$$

(see, e.g., [5]).

For the nodes (1.5) the projection A may be written in the following form ([7])

$$Af(t) = \frac{1}{n+1} \sum_{i=0}^{n+1}{}'' f(t_i)(-1)^i \left[\frac{t^2-1}{t-t_i} U_n(t) - T_{n+1}(t)\right] \tag{2.3}$$

(where T_l and U_l denote the Chebyshev polynomials of the first and second kinds, respectively).

3. The Connection between the Projections A and P

Recently [5] for the projections A and P the following result was obtained

$$Af(t) = Pf(t) - \sum_{i=0}^{n+1} \theta_i f(t_i) \sum_{j=0}^{n+1} \operatorname{sgn} \theta_j\, y_j(t) \tag{3.1}$$

where the numbers θ_i are given by (2.2).

Let the polynomials y_j be written in the following way

$$y_j(t) = \sum_{i=0}^{n} s_{ij} t^i \qquad (j = 0, 1,..., n+1).$$

Further let $\tau_j = s_{0j} = y_j(0)$. With the above notations we see that projection (1.1) is symmetric if

$$-t_i = t_{n+1-i} \qquad (i = 0, 1, \ldots, n+1) \tag{3.2}$$

and

$$\begin{aligned} s_{ij} &= s_{i,n+1-j} && \text{for} \quad i \text{ even}, \\ &= -s_{i,n+1-j} && \text{for} \quad i \text{ odd} \end{aligned} \tag{3.3}$$

$$(i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, n+1).$$

In virtue of (3.3) and the above notations we have

$$\tau_j = \tau_{n+1-j} \qquad (j = 0, 1, \ldots, n+1). \tag{3.4}$$

LEMMA 3.1. *If n is even ($n \geq 0$) and if the projection P is such that for it* (3.4) *holds, then $\Lambda_A(0) = \Lambda_P(0)$ where A is given by* (2.1), *P is given by* (1.1) (*for $m = n+1$*).

Proof. From (3.1) and (1.1) it follows that

$$\Lambda_A(0) = \sum_{i=0}^{n+1} \left| y_i(0) - \theta_i \sum_{j=0}^{n+1} \operatorname{sgn} \theta_j y_j(0) \right| = \sum_{i=0}^{n+1} \left| \tau_i - \theta_i \sum_{j=0}^{n+1} \operatorname{sgn} \theta_j \tau_j \right|.$$

From (2.2) we obtain $\operatorname{sgn} x_j = (-1)^{j+1}$. For n even it follows from (2.2) that $\operatorname{sgn} \theta_j = (-1)^j$ ($j = 0, 1, \ldots, n+1$). Hence and from (3.4) is

$$\sum_{j=0}^{n+1} \operatorname{sgn} \theta_j \tau_j = 0 \qquad \text{and now} \quad \Lambda_A(0) = \sum_{i=0}^{n+1} |\tau_i| = \Lambda_P(0).$$

The above lemma will be used in Section 4.

4. PROOF OF THEOREM 1.1

We prove first the left inequality of (1.7). For n even $T_{n+1}(0) = 0$ and $|U_n(0)| = 1$. Hence in virtue of Lemma 3.1 and (2.3) we have

$$\Lambda_P(0) = \Lambda_A(0) = (1/(n+1)) \sum_{i=0}^{n+1}{}'' 1/|t_i| = \lambda_n .$$

It is obvious that the inequality $\lambda_n \leq \|P\|$ holds for every projection satisfying the assumptions of the our theorem. In particular we have $\lambda_n \leq \|\bar{P}\|$

Now we prove the second inequality of (1.7). From (2.3) we see that projection A is a symmetric projection ($A \in \sigma$). From the definition of a minimal projection we have $\| \bar{P} \| \leqslant \| A \|$. In the recent paper [2] the authors proved that $\| A \| \leqslant \lambda_n + 1$. Hence the result.

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